# Effects of electron scattering on Cerenkov light output

SNO—STR—96—006

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14 February 1996

Abstract. Scattering of electrons in a medium such as water must affect the coherence of Cerenkov radiation emitted by successive track segments. I show that coherence between short segments is not significantly affected by scattering and that the light output from long segments is not significantly dependent on coherence between them. Scattering of electrons in water has negligible effect on the energy radiated as Cerenkov light.

## 1. Introduction

The cross section for coulomb scattering of charged particles is finite because of screening by atomic electrons. The mean distance between successive scatters is $\sim 1\mu$m for electrons in water, approximately twice the wavelength of visible Cerenkov radiation. This has led some people to worry that scattering may destroy the coherence between light emitted from successive segments of track and thereby suppress the energy radiated to levels substantially below those for straight tracks of the same length. In modelling Cerenkov radiation it is generally assumed that the light emitted from a segment of track is that which would be emitted by a segment of a very long straight track; it is important to establish the extent to which this assumption is valid. In this note I show that scattering does not destroy coherence until successive segments are so long that the degree of coherence between them has negligible effect on the total energy radiated. The amount of light lost because of scattering is certainly less than a few parts in one thousand.

My work is an extension of an earlier note by Komar [1] and like his is based on the treatment of Cerenkov radiation given in Schiff [2]. Amplitudes are determined by integration over the electron current density and there are no internal phases which might be reset by scattering.

## 2. Cerenkov light emitted by straight segments of track

After removal of factors irrelevant to the present problem, the amplitude for radiation emitted at an angle $\theta$ to the direction of a straight segment of track can be written as

$$A(\theta) = \frac{1}{2}\sin\theta \int e^{i\alpha z}dz \tag{1}$$

with

$$\alpha = \frac{2\pi}{\lambda}\left(\frac{c}{v} - n\cos\theta\right) \tag{2}$$

where $\lambda$ is the wavelength of the light, $v$ the speed of the electron and $n$ the refractive index of the medium. If the integral is taken from $-L/2$ to $+L/2$ the result is

$$A(\theta) = \sin\theta \frac{\sin\alpha\frac{L}{2}}{\alpha} \tag{3}$$

and the intensity is

$$I(\theta) = \sin^2\theta\left(\frac{\sin\alpha\frac{L}{2}}{\alpha}\right)^2 \tag{4}$$

The total light output for length L is given by

$$I(L) = \int_{-1}^{+1}\sin^2\theta\left(\frac{\sin\alpha\frac{L}{2}}{\alpha}\right)^2 d\cos\theta \tag{5}$$

The integral can be carried out exactly in the limit $L/\lambda \to \infty$ to yield the asymptotic light output

$$I_A(L) = \frac{L}{\lambda}\left(\frac{\lambda^2}{4n}\right)\sin^2\theta_c \tag{6}$$

where $\theta_c$ is the Cerenkov angle given by

$$\cos\theta_c = \frac{c}{nv} \tag{7}$$

For $L/\lambda$ large the total light output approaches the asymptotic value and the width of the Cerenkov cone is proportional to $L^{-1}$; a convenient measure is defined by the first order zeros of $\sin\alpha\frac{L}{2}$, which contain the principal maximum:

$$\begin{aligned}\pi\frac{L}{\lambda}\left(\frac{c}{v} - n\cos\theta\right) &= \pm\pi \\ \Delta\cos\theta &= \pm\frac{\lambda}{nL}\end{aligned} \tag{8}$$

$$\Delta\theta = \frac{\lambda}{L}\frac{2}{n\sin\theta_c} \tag{9}$$

where $\Delta\theta$ is the full width of the cone, centred on the opening angle $\theta_c$.

For $L/\lambda \to \infty$ the function $\left(\frac{\sin \alpha \frac{L}{2}}{\alpha}\right)^2$ is wholly contained by the limits of integration. As $L/\lambda$ is reduced this function spreads out and $I(L) < I_A(L)$. For $v = c$ in water ($n = 1.33$) one of the limits of (8) reaches $\cos\theta = 1$ at $L/\lambda = 3$ and for smaller values of $L/\lambda$ the width of the cone is bounded by $\cos\theta = 1$. Several examples of $I(\theta)$ (eq.(4)) are shown in fig.1 and in fig.2 the total light output and the width of the Cerenkov cone are plotted as a function of $L/\lambda$.

The following important points emerge from these figures. First, since the mean distance between scatters corresponds to $L/\lambda \sim 2$ for which light output is already 95% of the asymptotic value, if coherence were completely lost after each scatter the light output of the track would still be ~95% of asymptotic. Secondly, the width of the Cerenkov cone is very large for short segments. Most scatters are through very small angles (see section 5) and it is clear that while successive segments differ in angle by substantially less than the cone width coherence will be maintained. Scattering through 0.1 rad, for example, will have little effect for $L/\lambda < 10$, where the cone width exceeds 14°. For $L/\lambda = 10$ light output from such a segment is 99% of asymptotic.

## 3. Interference between colinear straight segments

The intensity from a single segment of length 2L is, from (4),

$$I(\theta) = \sin^2\theta \left(\frac{\sin \alpha L}{\alpha}\right)^2 \tag{10}$$

but it is instructive to break such a segment into two pieces. If the two pieces are each of length L, which maximises the significance of the interference term,

$$\begin{aligned} A(\theta) &= \frac{1}{2}\left[\int_{-L}^{0} e^{i\alpha z} dz + \int_{0}^{L} e^{i\alpha z} dz\right] \sin\theta \\ &= \left[e^{-i\alpha\frac{L}{2}} \sin\alpha\frac{L}{2} + e^{i\alpha\frac{L}{2}} \sin\alpha\frac{L}{2}\right] \sin\theta \end{aligned} \tag{11}$$

Taking the modulus squared of (11) yields for the intensity

$$I(\theta) = \left[\left(\frac{\sin \alpha\frac{L}{2}}{\alpha}\right)^2 + \left(\frac{\sin \alpha\frac{L}{2}}{\alpha}\right)^2 + 2\cos\alpha L \left(\frac{\sin \alpha\frac{L}{2}}{\alpha}\right)^2\right] \sin^2\theta \tag{12}$$

The first two terms in (12) are the intensities from the two pieces of length L and the last term is the contribution of interference between them. We can of course set $2\sin^2\alpha\frac{L}{2} = 1 - \cos\alpha L$ and obtain at once (10). The virtue of the decomposition is that it illuminates the underlying physics. If $L/\lambda$ is sufficiently large that the radiated energy is very close to the asymptotic limit,the integral over solid angle of the interference term in (12) must be negligible. The term $\cos(\alpha L)$ oscillates rapidly with $\cos\theta$ and the interference term averages to a very small number — $\alpha L$ varies from $-2\pi$ to $+2\pi$ over the width of the cone defined by (8).

In this limit the effect of the interference term is to narrow the width of the Cerenkov cone. If we have two contiguous segments, each of length L such that the Cerenkov light output is close to asymptotic, then if they make a straight line the cone is of width appropriate to 2L and the light output that for a straight line of length 2L. If there is a kink between the segments and the overlap of the narrow cones is negligible,then the interference term is negligible before integration over solid angle. There are then two cones,each of width appropriate to length L, and the total light output is equal to that for length 2L. Thus the total light output from two straight segments each of length $L/\lambda = 40$, for which $\Delta\theta = 3.3°$ and light output is 99.9% asymptotic, must have negligible dependence on the angle of the kink between them. Shorter segments are not so easily understood and demand numerical integration of the intensity.

## 4. Numerical calculations.

In the general case a track is kinked, the two segments are both below asymptotic light output and there is a substantial overlap of the Cerenkov cones. Fig.3 shows two straight segments of track, each of length L and separated by a distance $\mathcal{L}$.($\mathcal{L} = 0$ corresponds to contiguous segments.) For this case the intensity is given by

$$I(\theta) = \sin^2\theta\left(\frac{\sin\alpha\frac{L}{2}}{\alpha}\right)^2 + \sin^2\theta'\left(\frac{\sin\alpha'\frac{L}{2}}{\alpha'}\right)^2 + 2\mathcal{S}\left(\frac{\sin\alpha\frac{L}{2}}{\alpha}\right)\left(\frac{\sin\alpha'\frac{L}{2}}{\alpha'}\right)\cos\left\{\alpha\frac{L}{2} + \alpha'\frac{L}{2} + \alpha'\mathcal{L}\right\} \tag{13}$$

where

$$\alpha' = \frac{2\pi}{\lambda}\left(\frac{c}{v} - n\cos\theta'\right)$$

$$\cos\theta' = \cos\theta\cos\chi + \sin\theta\sin\phi\sin\chi$$

$$\mathcal{S} = \sin^2\theta\cos\chi - \sin\theta\cos\theta\sin\phi\sin\chi$$

The factor $\mathcal{S}$ is the angular dependence of the scalar products of the electric fields from the two segments.

In the limit $\chi \to 0$, $\mathcal{L} \to 0$ (13) reduces to (12). It is clear that the greater the value of $\mathcal{L}$ the faster the interference term oscillates: interference is most important between contiguous segments.

I have integrated (13) numerically for contiguous segments and integral values of $L/\lambda$ between 1 and 10. The effect of a kink is most pronounced for $L/\lambda = 1$. For this case light output falls below that for a straight segment of length $2\lambda$ by 0.1% for $\chi = 0.1$ rad, 0.5% for $\chi = 0.2$ rad and by 1% for $\chi = 0.3$ rad. For $L/\lambda \geq 2$ the light output falls below that for a straight segment of length 2L by less than 0.1% for $\chi = 0.2$ rad. Thus quite large kinks have a very small effect on the light output from short lengths of track. It remains only to discover whether coherence is lost before the light output becomes asymptotic.

## 5. Scattering probabilities

The mean distance between successive scatters in water is $\sim 1\mu$m ($2\lambda$ for $\lambda =$ 500 nm). (The exact number depends on fine details of the atomic screening and is of no importance here.) I assume that the relative probability of a single scatter deflecting an electron through an angle $> \theta$ is given by

$$P(>\theta) = \frac{\theta_1^2}{\theta^2 + \theta_1^2} \tag{14}$$

where

$$\theta_1 = \frac{Z^{\frac{1}{3}} mc^2}{137pc} \tag{15}$$

$Z$ is the atomic number of the scattering atom and $p$ the momentum of the electron. (The results (14) and (15) are taken from [3].)

The screening angle $\theta_1$ does not exceed 0.01 rad even for an electron kinetic energy as low as 0.5 MeV. The relative probability of single scattering through even 0.1 rad is therefore $< 0.01$. It is clear without further calculation that single scatters will not destroy the coherence of segments such that $L/\lambda \lesssim 40$ and so cannot affect the total light output of a track at any significant level. The burden is passed to multiple scattering.

Define a distance $L_c$ such that the width of the Cerenkov cone for a straight track is equal to the root mean square multiple scattering angle

$$\Delta\theta(L_c) = \langle\theta^2(L_c)\rangle^{\frac{1}{2}}$$

In the simple theory of multiple scattering [3]

$$\langle\theta^2(L)\rangle^{\frac{1}{2}} = \frac{E_s}{\beta cp}\sqrt{\frac{L}{X_0}} \quad (16)$$

where $E_s$ = 21 MeV and $X_0$ = 36 cm for water. With $pc$ in MeV and L in $\mu$m (16) becomes

$$\langle\theta^2(L)\rangle^{\frac{1}{2}} = \frac{0.035}{\beta cp}\sqrt{L}$$

From (9)

$$\Delta\theta = 2.28\frac{\lambda}{L}$$

and so

$$L_c = 16.2(\beta cp\lambda)^{\frac{2}{3}} \quad \mu\text{m}$$

The values of $L_c, L_c/\lambda$ for $\lambda = 0.5\mu$m and $\Delta\theta(L_c)$ are given in the table, for electron kinetic energies $T$ between 0.5 and 10 MeV. The values of $L_c$ for 0.5 and 1 MeV correspond to $\sim$ 10 individual scatters. In this region multiple scattering is a poor approximation; $< \theta^2 >^{1/2}$ is overestimated and so $L_c$ underestimated.

If coherence were destroyed after a distance $L_c$ the corresponding loss of light would be read from fig.2. For $L/\lambda = 20$, approximately the smallest values of $L_c$ in the table, the light output is 99.5% asymptotic. In fact numerical integration of (13) gives the result that for two segments each of length $20\lambda$, with a kink of 0.114 rad (corresponding to $\Delta\theta =< \theta^2 >^{\frac{1}{2}}$) the light output is 99.75% asymptotic; there exists a small positive interference term from the regions where the cones overlap. For a kink of 0.228 rad the output has only dropped to 99.73%. The assumption that segments of length $> L_c$ are incoherent is unduly pessimistic.

## 6. Conclusions

It is clear that the Cerenkov light output from electron tracks in water is not reduced by more than a few parts in one thousand as a result of electron scattering,even for kinetic energies as low as 0.5 MeV. The reason is that for short segments the individual Cerenkov cones are wide and scattering through any angle of significant probability barely affects the overlap of the cones. For long segments the interference term contributes negligibly to the total light output even if the segments are colinear.

It is entirely safe to use the asymptotic formulae for the total output of Cerenkov light from electron tracks in water; the output is proportional to the track length (but the constant of proportionality of course varies with the speed of the electron). The width of the Cerenkov cone from a segment of length $L > L_c$ will however be limited by $\Delta\theta(L_c)$ rather than by $\Delta\theta(L)$. This will have very little effect on the angular distribution of light from the whole track, which is dominated by the cumulative multiple scattering.

## References


[1] R J Komar unpublished note (undated)
[2] L I Schiff Quantum Mechanics (2nd ed., sect.57) McGraw-Hill (1955)
[3] B Rossi High Energy Particles Prentice-Hall (1952)


**Table**

| T (MeV) | $L_c$ ($\mu$m) | $L_c/\lambda$ | $\Delta\theta(L_c)$ |
|---|---|---|---|
| 10 | 48.3 | 96.6 | 1.35° |
| 5 | 31.7 | 63.4 | 2.06° |
| 2 | 18.3 | 36.6 | 3.57° |
| 1 | 12.4 | 24.8 | 5.27° |
| 0.5 | 8.4 | 16.8 | 7.78° |

## Figure captions

1. Relative intensities of Cerenkov radiation from short straight tracks, as a function of the cosine of the angle of emission. The nominal Cerenkov angle $\theta_c$ is indicated by a vertical arrow. The figures are taken from eqs(1-4) with $v = c$, $n = 1.33$.

2. The figure shows the light output per unit length of track as the fraction of the asymptotic value (solid curve). The results were obtained by numerical integration of eq.(5), with $v = c$, $n = 1.33$. Also shown is the angular width of the Cerenkov cone (broken curve,right hand scale). For $L/\lambda < 3$ the width of the cone is bounded by the limit of the physical region.

3. The figure defines quantities used in eq.(13). Two segments of track, each of length L, are separated by a distance $\mathcal{L}$. The second lies in the yz plane at an angle $\chi$ to the first.

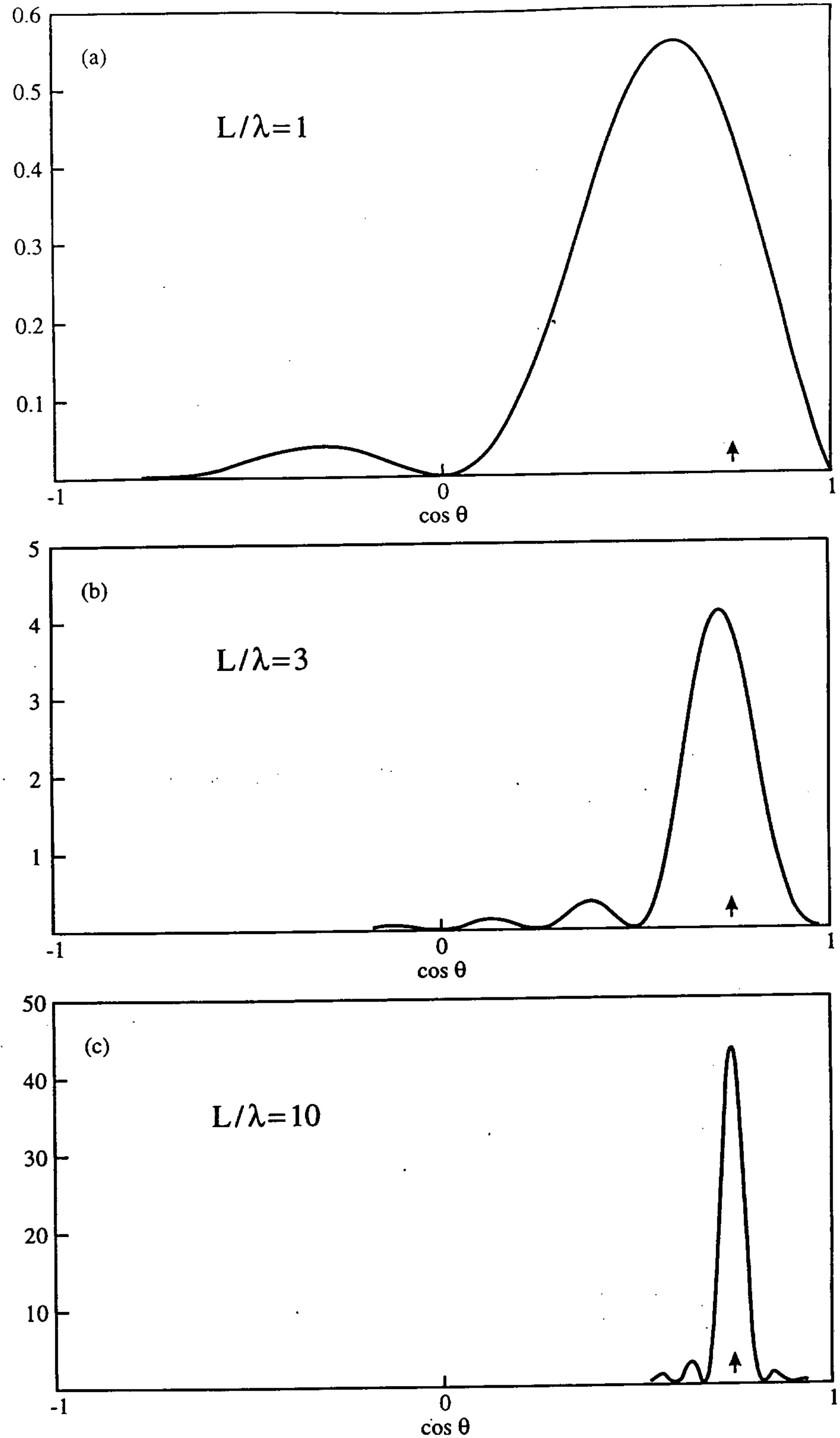
(a)
L/λ=1
0.6
0.5
0.4
0.3
0.2
0.1
-1
0
1
cos θ
(b)
L/λ=3
5
4
3
2
1
-1
0
1
cos θ
(c)
L/λ=10
50
40
30
20
10
-1
0
1
cos θ

Fig. 1

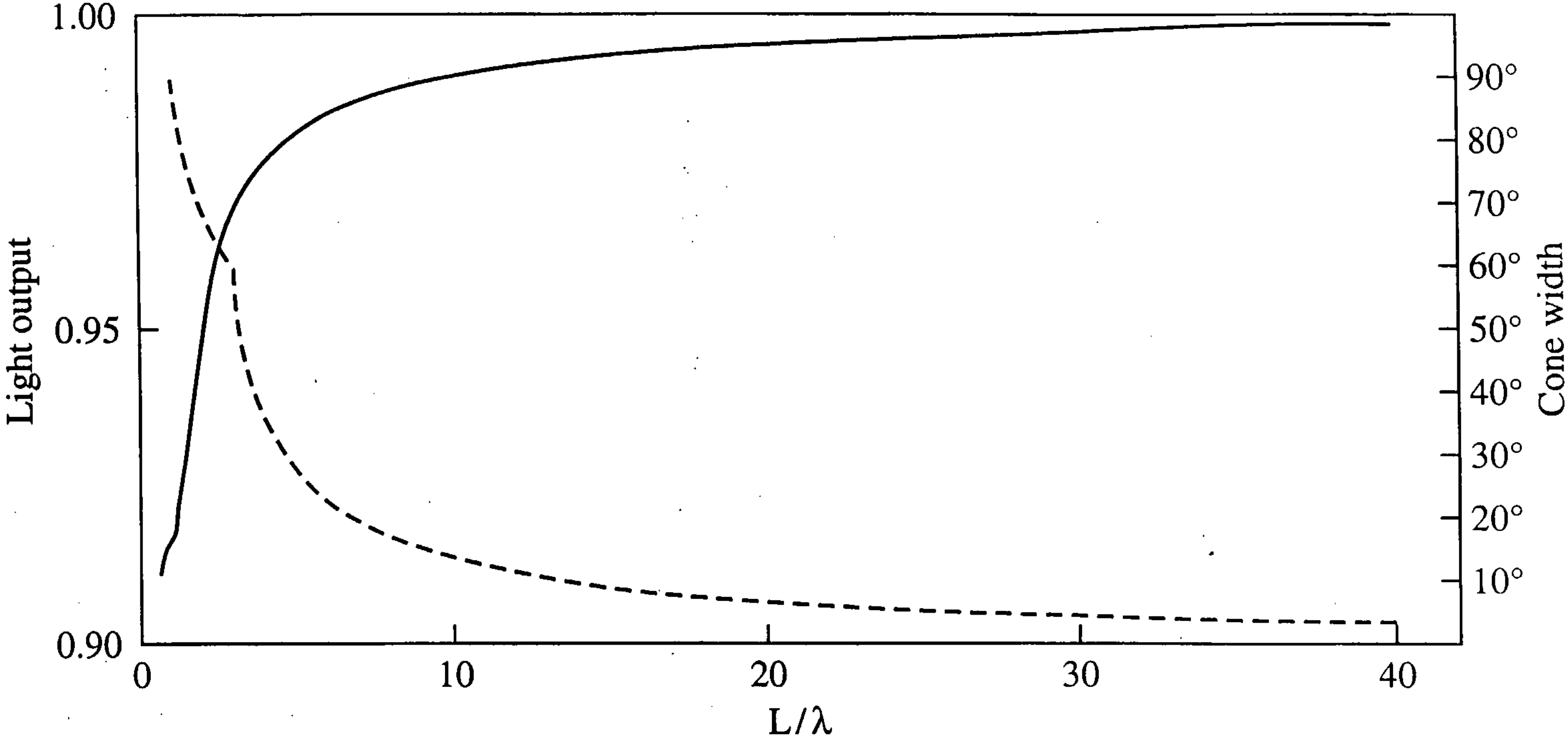
1.00
0.95
0.90
Light output
90°
80°
70°
60°
50°
40°
30°
20°
10°
Cone width
0
10
20
30
40
L/λ

Fig. 2

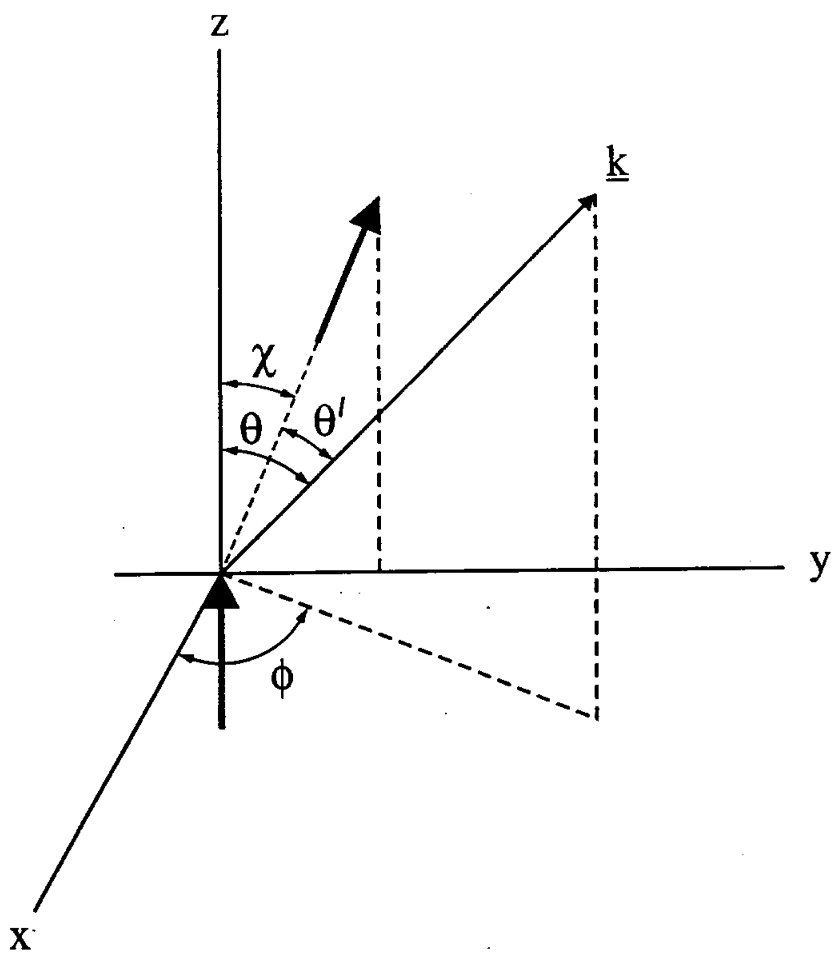


Fig. 3